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A METHOD *of expreſſing, when poſſible, the* VALUE *of* ONE VARIABLE QUANTITY *in* INTEGRAL POWERS *of* ANOTHER *and* CONSTANT QUANTITIES, *having given* EQUATIONS *expreſſing the* RELATION *of thoſe* VARIABLE QUANTITIES. *In which is contained the* GENERAL DOCTRINE *of* REVERSION *of* SERIES, *of* APPROXIMATING *to the* ROOTS *of* EQUATIONS, *and of the* SOLUTION *of* FLUXIONAL EQUATIONS *by* SERIES. *By the Rev.* J. BRINKLEY, *M. A. ANDREWS' Profeſſor of Aſtronomy, and M.R.I.A.*

THE moſt general and uſeful problem in analytics is, from a given relation between two variable quantities to expreſs one of thoſe quantities in terms of the other and conſtant quantities. The caſes however in which this can be completely performed are few in compariſon of thoſe in which it can be only partially done. Among the partial ſolutions are thoſe by ſeries not terminating. When ſuch ſeries converge they afford the ſolution required. Various methods have been given by authors for obtaining theſe ſeries principally derived from thoſe given by Sir I. Newton. Of theſe the method of aſſuming a ſeries with coefficients to be determined

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termined from a compariſon of homologous terms is, perhaps, the beſt where it can be practiſed; yet the caſes are very numerous where without other aſſiſtance it is difficult and almoſt impoſſible to practice it with any advantage. A method, therefore, which beſides being in all caſes as ſimple as any of the others is as general as can be deſired, and is often attended with the ſuperior advantage of demonſtrating the law of the ſeries, muſt be an object for the conſideration of mathematicians. Such a method is attempted in the following pages. Its foundation is built upon a theorem firſt given by that excellent mathematician Dr. Brooke Taylor. This theorem, given in Cor. 2. Prop. 7. page 23, of his method of Increments, is well known, and is in purport as follows:

If x and z be two variable quantities, the relation of which is given, then while x by flowing uniformly is increaſed by $\dot{x}$, z will be increaſed by $\dot{z} + \frac{\ddot{z}}{1.\ 2.} + \frac{\overset{3}{\dot{z}}}{1.\ 2.\ 3.} +$ &c. In which the values of $\dot{z}$, $\ddot{z}$, &c. are to be determined from the given equation.

It readily occurs that this theorem contains a method of deriving the values of one quantity by a ſeries aſcending by powers of the other: and accordingly ſome authors have uſed it in a few ſimple caſes, but have not attempted a general uſe. And upon conſideration it is obvious that without farther aſſiſtance it cannot be practiſed in caſes at all complex. For if in an equation expreſſing the relation of x and z the ſucceſſive fluxions be derived one from the other generally and without having regard to particular values of x and z, almoſt inſuperable trouble would ariſe except in the moſt ſimple caſes, and the method be very far inferior to others. This farther aſſiſtance I have endeavoured to give in the following pages, principally by theorems for taking fluxions of different orders per ſaltum, that is, without finding the fluxions of the inferior orders. Theſe will render the theorem of Taylor of the moſt extenſive utility, as will beſt be ſeen by the examples hereafter given.

M. De la Grange is the only author I know of who has attempted to ſimplify the computation of $\dot{z}$, $\ddot{z}$, &c. This he has done by a moſt elegant theorem for an equation of a particular form (See Couſin's " Aſtron. Phyſique, Art. 20, p. 15.") But no uſe can be made of this theorem except in equations of that particular form. The theorems for taking fluxions per ſaltum will enable us to compute the values of $\dot{z}$, $\ddot{z}$, &c. by ſubſtituting the values of x and z when they begin to flow, and as in that caſe it often happens that the problem is ſuch that x and z begin from nothing, the concluſions are then derived in the moſt ſimple manner.

The method of aſſuming a ſeries with undetermined coefficients for the quantity to be found, beſides the objections in every particular caſe to the legality of ſuch an aſſumption not being ſelf-evident, often requires perplexing conſiderations to avoid introducing unneceſſary terms. Indeed the greateſt difficulty often occurs in that part. In this method no ſeries is to be aſſumed. The ſeries derived follows from the nature of the problem. Oftentimes its law even in very complex caſes can be derived, in which by the method of aſſuming a ſeries it would be almoſt impoſſible to demonſtrate it. Thus the truth of the law of the Multinomial Theorem, when the power is negative or fractional, is demonſtrated by this method. It was done by De Moivre for integral powers, and I know of no author who has generally demonſtrated it for all powers. The examples given to illuſtrate the method are moſt of them ſuch as are well known, and may be compared with the ſame as done by other methods. Among them are two ſeries firſt given by Mr. James Gregory (See Comm. Epiſt.) the inveſtigation of the latter of which has been conſidered by mathematicians as very difficult.

Demonſtration of Dr. Brooke Taylor's Theorem*.

Theo. If z and x be cotemporaneous values of two quantities any how related, and z and $\dot{x}$ = flux. of x, cotemporaneous increments, of which x is uniformly generated, then will

$$z + \underset{\cdot}{z} = z + \frac{\dot{z}}{1} + \frac{\ddot{z}}{1.\ 2.} + \frac{\overset{\because}{z}}{1.\ 2.\ 3.} + \quad - \quad - \frac{\overset{\overset{m}{\cdot}}{z}}{1.\ 2.\ldots m} +,\ \text{\&c.}$$

when this ſeries terminates or converges.

Demonstration.

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* Authors who have given this theorem have not been ſo attentive to accuracy of demonſtration as the importance of the theorem ſeems to require.

DEMONSTRATION.

$$\text{LET}\quad \left.\begin{array}{llllll} x, & x+\underset{\cdot}{x}, & x+2\underset{\cdot}{x} & - & - & ,\ x+\dot{x} \\ z, & z', & z'', & \overset{3}{z'},\ \overset{4}{z'} & & -\ z+\underset{\cdot}{z} \end{array}\right\}\begin{array}{l}\text{be cotemporaneous} \\ \text{values of } x \text{ and } z\end{array}$$

$$\text{LET alſo}\quad \left.\begin{array}{lllll} a, & a', & a'', & \overset{3}{a'} & \&c. \\ b, & b', & b'', & \&c. & \\ c, & c'', & \&c. & & \\ d, & \&c. & & & \end{array}\right\}\begin{array}{l}\text{be differences of the reſpec-} \\ \text{tive orders.}\end{array}$$

THEN by the theorem for differences.

$z + \underset{\cdot}{z} = z + na + \frac{n.\ \overline{n-1}}{1.\ 2.}\, b + \frac{n.\ \overline{n-1}.\ \overline{n-2}}{1.\ 2.\ 3}\, c +$, &c. where n is the number of ſucceſſive values from x to $x + \dot{x}$, or from z to $z + \underset{\cdot}{z}$. Now if n be increaſed *ſine limite*, any aſſigned number of terms of this quantity approaches to the ſame number of terms in the ſeries,

$z + na + \frac{n^2\ b}{1.\ 2.} + \frac{n^3\ c}{1.\ 2.\ 3.} +$, &c. as its limit, or becauſe $n = \frac{\dot{x}}{\underset{\cdot}{x}}$, to its equal $z + \frac{a}{\underset{\cdot}{x}}\dot{x} + \frac{b}{\underset{\cdot}{x}^2} \times \frac{\dot{x}^2}{1.\ 2.} +$, &c. But when n is ſo increaſed the limiting ratio of $a : x$ or the limiting ratio of the increments of z and x is the ratio of the fluxions of z and x, and it follows therefore that when n is increaſed *ſine limite*, the limiting value of $\frac{a}{\underset{\cdot}{x}} = \frac{\dot{z}}{\dot{x}}$. Alſo for the ſame reaſon $\frac{b}{\underset{\cdot}{x}} = \frac{\dot{a}}{\dot{x}}$, $\frac{c}{\underset{\cdot}{x}} = \frac{\dot{b}}{\dot{x}}$, &c. Whence the limiting value of $\frac{b}{\underset{\cdot}{x}^2} = \frac{\dot{a}}{x\underset{\cdot}{x}} = \frac{\ddot{z}}{\dot{x}^2}$, becauſe $\frac{\dot{a}}{\underset{\cdot}{x}} = \frac{\ddot{z}}{\dot{x}}$

of $\frac{c}{\underset{\cdot}{x}^3} = \frac{\dot{b}}{\dot{x}\underset{\cdot}{x}^2} = \frac{\overset{3}{\dot{z}}}{x^3}$, becauſe $\frac{\dot{b}}{\underset{\cdot}{x}^2} = \frac{\overset{3}{\dot{z}}}{\dot{x}^2}$,

&c. &c.

Whence the limiting value of $z + n\,a + \frac{n.\overline{n-1}}{1.\;2.}\,b + \frac{n.\overline{n-1}.\overline{n-2}}{1.\;2.\;3}\,c$ + &c. when n is increaſed *ſine limite* is $z + \frac{\dot{z}}{1} + \frac{\ddot{z}}{1.\;2.}$ +, &c. And becauſe when the former ſeries terminates its value is $z + \underset{\cdot}{z}$: and when it converges its limit is alſo $z + \underset{\cdot}{z}$ $\because z + \underset{\cdot}{z} = z + \frac{\dot{z}}{1} + \frac{\ddot{z}}{1.\;2.}$, + &c. when the ſeries terminates or converges. When it does not converge, nothing can be aſſerted of it, becauſe we cannot reaſon concerning a limit which does not exiſt.

Problems

Problems for finding Fluxions *per Saltum.*

PROB. 1. To find the n^{th} fluxion of x^m when x does not flow uniformly, and m denotes any whole, fractional or negative number.

SOLUTION. Let a, b, c, d - - $\varkappa$ - - δ, γ, β, α be the 1ſt, 2d, 3d, 4th, - - kth - $n-4$, $n-3$, $n-2$, $n-1$ fluxions of x.

THEN the n^{th} fluxion of x^m =

$$m x^{m-1}\left.\begin{array}{l}\overset{n}{\dot{x}} + n.a\alpha\\ \frac{n.n-1}{1.2}b\beta\\ \frac{n.n-1.n-2}{1.2.3}c\gamma\\ \&c.\end{array}\right| m.\overline{m-1}\,x^{m-2}\left.\begin{array}{l}\frac{n.n-1.n-2}{1.2.1}ab\gamma\\ +\frac{n.n-1.n-2.n-3}{1.3.2.1}ac\delta\\ \&c.\end{array}\right| m.\overline{m-1}.\overline{m-2}\,x^{m-3}+, \&c.$$

THE following are the laws of this ſeries:

1. THE index of x diminiſhes in each term by unity, and is to be continued till it becomes o or $m-n$.

2. THE coefficient of x^{m-v} is the product of $m.\overline{m-1}.$ - - $\overline{m-v-1}$, into the ſum of quantities, with numeral coefficients annexed, deduced from the different fluxions of x.

3. THESE quantities are formed by multiplying together a number v of the ſeveral fluxions of x, ſo that the ſum of their exponents ſhall be n. Thus if $\overset{p}{a}\,\overset{q}{b}\,\overset{r}{c}\,\overset{s}{d}$ be one of theſe quantities $p + 2q + 3r + 4s = n$ and $p + q + r + s = v$.

NOTE—By exponent of a fluxion is meant its order. Thus the exponent of d or of the fourth fluxion of x is 4.

4. To the quantity $\overset{p}{a}\,\overset{q}{b}\,\overset{r}{c}\,\overset{s}{d}$ - - $\overset{\sigma}{x}$ is to be annexed, for a coefficient, a fraction the numerator of which is $n.\overline{n-1}.\overline{n-2}$ - $\overline{k+1}$, and the denominator $p \times \overline{p-1}.$ - - $1 \times \overset{q}{2}.\,q.\overline{q-1}$ - $1 \times \overset{r}{\overline{3\ 2}}.\,r.\overline{r-1}$ - - $1 \times \overset{s}{\overline{4.\ 3.\ 2}} \times s.\overline{s-1}$ - - $1 \times \overset{\sigma-1}{\overline{k\ k-1\ \text{- -}\ 1}} \times \sigma \times \overline{\sigma-1}$ - - 1. The law of continuation of which is evident*.

The Demonſtration, as far as regards the 1ſt, 2d and 3d laws of the ſeries, is readily deduced from conſidering the manner in which the ſucceſſive fluxions of $\overset{m}{x}$ are derived. The demonſtration of the fourth law is ſomewhat more difficult, but may be deduced as follows: A quantity $\overset{p}{a}\,\overset{q}{b}$ prefixed to a power of x is evidently derived by taken the fluxion of x, $\overline{p+q}$ times, and of a, q times in every different order, with the exception that each a muſt be taken before the b, which is derived from it. Conſequently the coefficient of $\overset{p}{a}\,\overset{q}{b}$ muſt be the number of theſe different orders. This coefficient may therefore be deduced either from the doctrine of permutations or from that of probabilities. The former method is certainly the moſt natural, and at firſt ſight may appear ſhorter: but the latter is more readily applicable to general expreſſions. And from it the coefficient of $\overset{p}{a}\,\overset{q}{b}$ is deduced by finding the probability of taking a, a, a, - (p) $\underset{'}{a}$, $\underset{''}{a}$, $\underset{'''}{a}$ - (p) $\underset{'}{b}$, $\underset{''}{b}$, $\underset{'''}{b}$ - (q terms) in the order in which they are written. The marks underneath ſhewing the a's from which the correſponding b's are derived. The inverſe of the fraction expreſſing this probability is the number of different orders, and conſequently the coefficient of $\overset{p}{a}\,\overset{q}{b}$. The prob. that an a, from which a b is not derived is taken firſt is $\frac{p}{n}$, that another a of the ſame deſcription is taken next is $\frac{p-1}{n-1}$, &c. ſo that the probability that all the a's of that deſcription are taken previouſly to any of the a's, from whence the b's are derived is $\frac{p}{n} \times \frac{p-1}{n-1} \times \frac{p-2}{n-2} \times$ - - $\frac{1}{n-p-1}$. That an a is taken next is certainty or

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* Since writing the above I find that Dr. Waring, at the end of his "Meditationes Analyticæ," ſpeaking of "methodus deductionis & reductionis," mentions this problem, and gives the three firſt laws, in which indeed there is no difficulty; the fourth, the only one difficult to inveſtigate, he does not give, nor does he mention any uſe to which the problem may be applied.

$\frac{2q}{n-p}$. The Prob. that another a is taken next is $2 \times \frac{q-1}{n-p-1}$ becauſe each a, beſides its own chance, has the chance of the b, which is derived from it, &c. &c. Whence it follows that the probability that all the a^s will be taken before any of the b^s is $\frac{p}{n} \times \frac{p-1}{n-1} \times \frac{p-2}{n-2} \text{ - - } \times \frac{1}{\overline{n-p-1}} \times \frac{2q}{n-p} \times \frac{2 \times \overline{q-1}}{n-p-1} \times \frac{2 \times \overline{q-2}}{n-p-2} \times \text{ - - } \frac{2}{\overline{n-p+q-1}}$. The probability that the b derived from the firſt a is taken next is $\frac{1}{\overline{n-p+q}}$, &c.

Whence the prob. that the whole will be taken in the order in which they are written is

$$\frac{p \times \overline{p-1} \text{ - - } 1 \times 2^{q-1} \times q \times \overline{q-1} \times \overline{q-2} \text{ - - } 1}{n.\overline{n-1} \text{ - - - } \overline{n-p+2q-3}}.$$

The reciprocal of which fraction is the coefficient of $a^p b^q$. And by the ſame proceſs the general coeff. of $a^p b^q c^r d^s \text{ - - } x^\sigma$ as given in the 4th law is readily deducible.

The dem. by the method of permutations is conciſely as follows. If the quantities a, a, a, (p) a, b, a, b $(2q)$ were all different, the number of orders is $n.\overline{n-1}. \text{ - - } 1$, but as p quantities are the ſame, the number muſt be reduced by dividing by $p.\overline{p-1} \text{ - - } 1$, or the number of permutations of p things, and becauſe the permutations of two things are two without regard to order, when the order is fixed the whole number of permutations muſt be alſo divided by the number of permutations in each order that is fixed, that is by $2 \times 2 \times 2 \times$ &c. $(q) = 2^q$, alſo becauſe q a^s are the ſame, it muſt be divided by $q \times \overline{q-1} \times - - 1$. Whence the number of permutations or the coefficient of $a^p\, b^q$ is as above ſtated, &c. &c.

EXAMPLE I. The 6th fluxion of $x^m =$

$$= m\overset{6}{\dot{x}}\,x^{m-1} + \left.\begin{array}{r} 6\,\dot{x}\,\overset{5}{x} \\ 15\,\dot{x}\,\overset{4}{x} \\ 10\,\overset{3}{\dot{x}}{}^2 \end{array}\right| m.\overline{m-1}\,x^{m-2} + \left.\begin{array}{r} 15\,\dot{x}^2\,\overset{4}{\dot{x}} \\ 15\,\overset{2}{\dot{x}}{}^3 \\ 60\,\dot{x}\,\dot{x}\,\overset{3}{\dot{x}} \end{array}\right| m\ldots\overline{m-2}\,x^{m-3} + \left.\begin{array}{r} 20\,\dot{x}^3\,\overset{3}{\dot{x}} \\ 45\,\dot{x}^2\,\overset{2}{\dot{x}}{}^2 \end{array}\right| m\ldots\overline{m-3}\,x^{m-4} +$$

$$15\,\dot{x}^4\,\ddot{x}\,\big|\,m\ldots.\overline{m-4}\,x^{m-5} + m. - - \overline{m-5}\,\dot{x}^6\,x^{m-6}$$

EXAMPLE II. The 8th fluxion of x^2 when $x = o$, and the 1ſt, 3d, 5th, and 7th, or the fluxions of the uneven orders are alſo $= o$ is $56\,\overset{2}{\dot{x}}\,\overset{6}{\dot{x}} + 70\,\overset{4}{\dot{x}}{}^2$. For in this caſe the coeff. of $\overset{2}{\dot{x}}\,\overset{6}{\dot{x}} = \frac{8.\overline{7}}{2.1.1.} \times 2 =$ 56, and the coeff. of $\overset{4}{\dot{x}}{}^2 = \frac{8.7.6.5}{\overline{4.3.2.1}\,|\,\overline{2-1} \times 2.1} \times 2 = 70.$

EXAMPLE III. The coeff. of x^5 in the 7th fluxion of x_8, when the even fluxions are $= o$ is

$$\left.\begin{array}{l}\dfrac{7.6.5.4}{1.\overline{3.2.1}\,.\,2}{}_{2-1}.\ \dot{x}\,\overset{3}{\dot{x}}{}^{2}\\[2ex] \dfrac{7.6}{2.1.1}.\ \dot{x}^{2}\,\overset{5}{\dot{x}}\end{array}\right|\,8.\,7.\,6 = \overline{70\,\dot{x}\,\overset{3}{\dot{x}} + 21\,x^{2}\,\overset{5}{\dot{x}}}\times 8.\,7.\,6.$$

Cor. If $a^{\backprime} = a$, $b^{\backprime} = \dfrac{b}{1.\,2}$, $c^{\backprime} = \dfrac{c}{1.2.3}$ &c. The denominator of the coefficient to be affixed to $\overset{p}{a^{\backprime}}\,\overset{q}{b^{\backprime}}\,\overset{r}{c^{\backprime}}$ - - $\overset{\sigma}{x^{\backprime}}$ is $p.\overline{p-1}$ - - 1. $q.\overline{q-1}.$ - - 1 - - - $\sigma\times\overline{\sigma-1}$ - - 1 and the numerator $n.\,\overline{n-1}.\,\overline{n-2}$ - - - 1.

Problem 2. To find the n^{th} fluxion of xyz (m quantities.)

Solution. The n^{th} fluxion of xyz (m quantities).

$\overset{n}{\dot{x}}yz$, &c. + $x\overset{n}{\dot{y}}z$, &c. + $xy\overset{n}{\dot{z}}$, &c.

$+\,n\,\overset{n-1}{\dot{x}}\,\dot{y}z$, &c. $+\,n\,\dot{x}\overset{n-1}{\dot{y}}\,z$ &c. + &c.

$+\,n.\overline{n-1}\,\overset{m-2}{\dot{x}}\,\dot{y}\dot{z}$, &c. + &c. To form this quantity the ſum of &c.

all the $\overset{\alpha}{\dot{x}}\overset{\beta}{\dot{y}}\overset{\gamma}{\dot{z}}$, &c. muſt be taken where $\alpha + \beta + \gamma +$, &c. $= n$. Affixing when α or β, or γ, &c. $= o$, inſtead of $\overset{\alpha}{\dot{x}}$, x, inſtead of $\overset{\beta}{\dot{y}}$, of $\overset{\beta}{\dot{y}}, y$, &c. The coefficient of $\overset{\alpha}{\dot{x}}\overset{\beta}{\dot{y}}\overset{\gamma}{\dot{z}}$, &c. is readily deducible by the methods in the former problem, and is =

$$\frac{n.\overline{n-1}.\overline{n-2}\;-\;-\;-\;-\;1}{\alpha\times\overline{\alpha-1}\;-\;-\;1\times\beta\times\overline{\beta-1}\;-\;-\;-\;1\times\gamma\times\overline{\gamma-1}\times\;-\;-\;-\;1\times,\ \&c.}$$

Problem 3. To find the n^{th} fluxion of the ſine of an arch x taken m times when the arch does not flow uniformly.

Solution. Radius being unity. The n^{th} fluxion of the ſine of mx

$$= m\overset{n}{\dot{x}}\ cs\,mx\ \left.\begin{array}{l} -\,n\dot{x}\,\dot{x}^{n-1} \\ -\,\frac{n.\overline{n-1}}{1.2}\ddot{x}\,\dot{x}^{n-2} \\ -\,\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}\overset{3}{\dot{x}}\,\dot{x}^{m-3} \\ \quad\&c. \end{array}\right|\ m^2\,s,\ mx\ —,\ \&c.$$

The following are the laws of this ſeries:

1. The quantities to which the products of the fluxions and their coefficients are affixed are ſucceſſively $m\,cs$, mx: $m^2\,s$, mx: $m^3\,cs$, mx: &c. The ſign is + or — according as the number of preceding terms of coſines is even or odd.

2. THE number of fluxional factors to be affixed to the rth term is r, and the ſum of their exponents is to be n. Thus if $\dot{x}^{p} \times \overset{2}{\dot{x}}{}^{q} \times$ &c. be one of theſe products $p + 2q +$ &c. $= n$, and $p + q +$ &c. $= r$.

3. THE coefficient of $\dot{x}^{p} \times \overset{2}{\dot{x}}{}^{q}$, &c. is as ſtated in Prob. I.

EXAMPLE. The fourth fluxion of the ſine of $3x$, when $x = 0$ is $3\overset{4}{\dot{x}} - \overset{4}{3}.\ 2.\ \dot{x}^{2}\ \overset{2}{\dot{x}}$.

COR. The n^{th} fluxion of the coſine of mx is had by ſubſtituting in the above ſeries for the coſine of mx, $-s$, of mx, and for s, mx, cs, mx.

The Application of the preceding Problems.

PROBLEM 5. The relation of two quantities being expreſſed by one or more equations to find the value of one of them in a ſeries aſcending by integral powers of the other.

SOLUTION. Let x and y be the two quantities to find x in a ſeries, aſcending by integral powers of y. Compute from the given equations, by help of the preceding problems, the values of x (A), $\dot{x}$ (B), $\ddot{x}$ (C), &c. when $y =$ a given value as a and $\dot{y} = y - a$ making y flow

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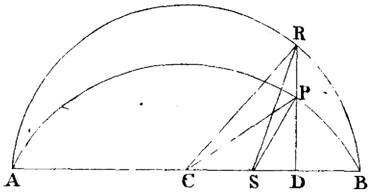

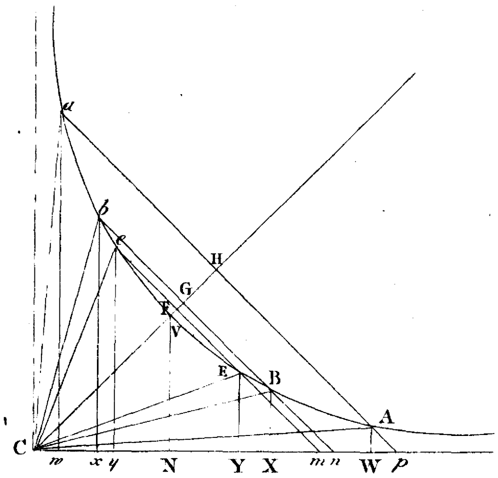

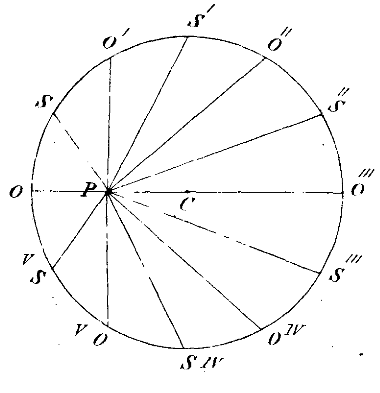

flow uniformly. Then by Taylor's theorem whilſt y changes its value from a to y, x will from A become $A + \frac{B}{1} + \frac{C}{1.2} + \frac{D}{1.2.3}$, &c.

For more readily uſing the preceding problems, it will generally be of uſe to clear the given equations from fractions, ſurds, &c. and ſometimes alſo to take the 2d, &c. fluxions generally, in order to have a more convenient equation, from which the particular fluxions of the higher orders are to be deduced. The particular fluxions of the different orders are to be taken *per ſaltum* by the preceding problems, ſubſtituting at the ſame time whenever convenient the values of x, $\dot{x}$, $\ddot{x}$, &c. previouſly found.

The utility and practice of this method will beſt appear by examples.

Example I. From the cubic equation $x^3 + qx + r = 0$, to deduce the values of x in a ſeries aſcending by the powers of r.

Solution. Let the ſucceſſive fluxions of this equation be taken by Cor. Prob. 1. making $a = \dot{x}$, $b = \frac{\ddot{x}}{1.2}$, $c = \frac{\dddot{x}}{1.2.3}$ &c. and $\dot{r}$ conſtant.

1. $3x$

$$1^{st}.\ \overline{3x^2+q}.\ a+r=0.$$

$$2^{d}.\ \overline{3x^2+q}.\ 1.2.b+a^2.3.2.x=0.$$

$$3^{d}.\ \overline{3x^2+q}.\ 1.2.3c+3.2.1.ab.3.2.x+a^3.3.2=0.$$

$$4^{th}.\ \overline{3x^2+q}.\ 1.2.3.4d+\left.\begin{array}{r}4.3.2.1\ ac\\ 4.3\qquad b^2\end{array}\right|3.2x+4.3.a^2b.3.2=0.$$

$$5^{th}.\ \overline{3x^2+q}.\ 1.2.3.4.5e+\left.\begin{array}{r}5.4.3.2.1\ ad\\ 5.4.3.2.1\ bc\end{array}\right|3.2x+\left.\begin{array}{r}5.4.3\ a^2c\\ 5.4.3\ ab^2\end{array}\right|3.2=0$$

&c. &c. &c.

$$\text{or } a=-\frac{r}{3x^2+q},\ b=-\frac{3a^2x}{3x^2+q},\ c=-\frac{3.2abx+a^3}{3x^2+q},\ d=-\frac{3.2acx+3b^2}{3x^2+q},\ e=-\frac{\overline{ad+bc}.\,3.2.x+3.\overline{a^2c+ab^2}}{3x^2+q}$$

&c. &c.

Calling the exponents of the quantities a, b, c, &c. their places in the ſeries, and the exponents of $\overset{p}{a}$, p: of $\overset{p}{b}$, $2p$, &c. the law of continuation is eaſily had. For the numerator of the quantity, the exponent of which is m, conſiſts of two terms, the firſt of which is $3.2.x$ into a coeff. which is the ſum of the products of every two quantities, the ſum of the exponents of which is m, and when m is even, $\frac{1}{2}$ the ſquare of the quantity, the exponent of which is $\frac{1}{2}m$ is to be added. The ſecond term is the ſum of all the quantities $3\alpha^2\beta$, $3.2\gamma\delta\varepsilon$, ſo that the ſum of the exponents of each quantity $= m$: and when m is a multiple of 3, the cube of the term, the exponent of which is $\frac{m}{2}$ is to be added.

Now when $r = o$, $x^3 + qx = o$, and the values of x are o, $+\sqrt{-q}$, ſubſtituting theſe values in the values of a, b, c, &c. found above, and r for $\dot{r}$, we have the three values of $a + b + c +$ &c. $= \dot{x} + \frac{\ddot{x}}{2} +$ &c. the three increments of x, while r from o becomes r. Let theſe values be A, B, C, and the values of x are A, $\sqrt{-q} + \mathrm{B}$, $-\sqrt{-q} + \mathrm{C}$.

The preceding is given as an example of the method, and not to ſhew its ſuperiority to others. Since by aſſuming a ſeries for x, and making uſe of the multinomial theorem, the ſame concluſion will be derived by a proceſs equally ſhort. Yet it muſt be obſerved, that the multinomial theorem is only a particular theorem far leſs extenſive indeed in its uſes than the method here given, and not at all more ready in practice.

Example II. Given the ſine of an arch (A) to find the ſine of n times that arch.

Solution. Let x = ſine of A, y = ſine of n A. Then $\frac{\dot{y}}{\sqrt{1-y^2}} = \frac{n\dot{x}}{\sqrt{1-x^2}}$ or $\dot{y}_2 \times \overline{1-x^2} = n^2\dot{x}^2 \times \overline{1-y^2}$, taking the fluxions generally making x flow uniformly, and dividing by $\dot{y}$.

$$\ddot{y} \times \overline{1-x^2} - \dot{y}x\dot{x} = -n^2\dot{x}^2 y.$$

The $\overline{m-2}$ fluxion of this equation being taken by Prob. 1 and 2, when $x = 0$ and $y = 0$,

$$\overset{m}{\dot{y}} + \frac{\overline{m-2}.\overline{m-3}}{2}\overset{m-2}{\dot{y}} \times -2\dot{x}^2 - \overline{m-2}\overset{m-2}{\dot{y}}\dot{x}^2 = -n^2\dot{x}^2\overset{m-2}{\dot{y}}$$

or $\overset{m}{\dot{y}} = -\dot{x}^2\overset{m-2}{\dot{y}} \times \overline{n^2 - \overline{m-2}|^2}$.

Now becauſe when x and $y = 0$, $\dot{y} = n\dot{x}$ and $\ddot{y} = 0$; it follows therefore that all the even fluxions of y are $= 0$; and taking for m the odd numbers 3, 5, 7, &c. and x for $\dot{x}$, we have $y = \dot{y} + \frac{\ddot{y}}{1.2} + \text{\&c.} = nx - \frac{n^2-1}{2.3}ax^2 + \frac{n^2-3^2}{4.5}bx^2 - \text{\&c.}$ a, b, &c. being the preceding terms. Alſo if k and l be the $p-1$, and p terms $l =$ (becauſe $m = \overline{2p-1}$) $= -kx^2 \times \frac{n^2 - \overline{2p-3}|^2}{\overline{2p-2} \times \overline{2p-1}}$.

The above ſolution affords a conſpicuous inſtance of the advantage of this method, in the ready manner in which the general law of the ſeries is derived. This ſeries has been inveſtigated by ſeveral authors ſince Sir I. Newton, who firſt invented it. But all have only deduced a few of the firſt terms, without any proof whatever of the law of the ſeries. Indeed to have deduced by any of their methods even the 10th term would have been an almoſt inſuperable labour.

Example III. To expreſs the hyperbolic logarithmic ſecant by a ſeries aſcending by powers of the arch.

Solution. Let a, s, and l be the arc, ſecant, and logarithmic ſecant, rad. being unity. By the nature of the circle

$\dot{a} = \frac{\dot{s}}{s\sqrt{s^2 - 1}}$, and alſo $\dot{l} = \frac{\dot{s}}{s}$

$\because \dot{l}^2 = \dot{a}^2 \times \overline{s^2 - 1}$, or taking the fluxions and making a conſtant,

$2\dot{l}\ddot{l} = \dot{a}^2 \times 2s\dot{s} = \dot{a}^2 \times 2s^2 \times \dot{l}$ or $\ddot{l} = \dot{a}^2 s^2 = \dot{l}^2 + \dot{a}^2$ (A).
But when $a = o$, $s = 1$, $\because l = o$ and $\dot{l} = \dot{a}\sqrt{s^2 - 1} = o \therefore$ whence from the equation A it follows that all the uneven fluxions of l are $= o$, becauſe any odd fluxion of the equation muſt contain in each term the inferior odd fluxions of l. For the conveniency

of applying Prob. 1. let $x = \dot{l}$, and (A) $\ddot{l} = \dot{a}^2 + x^2$, taking the $\overline{n-2}$ fluxion of this equation by Prob. 1. $\overset{n}{\dot{l}} = 2\dot{x}\overset{n-3}{\dot{x}}.\ \overline{n-2} + 2\overset{3}{\dot{x}}\overset{n-5}{\dot{x}}\ \frac{\overline{n-2}.\ \overline{n-3}.\ \overline{n-4}}{3.\quad 2.\quad 1} + 2\overset{5}{\dot{x}}\overset{n-1}{\dot{x}}\ \frac{\overline{n-2}\ -\ -\ \overline{n-6}}{5\ 4\ -\ -\ 1} + \&c.$

ſubſtituting for the fluxions of x, $\ddot{l}$, $\overset{3}{\dot{l}}$, &c. and dividing by 1, 2 - - n we get the general equation

$$\frac{\overset{n}{\dot{l}}}{1\ -\ -\ n} = 2.\ 2\frac{\ddot{l}}{1.\ 2}\times\frac{\overset{n-2}{\dot{l}}}{1.\ 2\ -\ \overline{n-2}}\times\frac{\overline{n-2}}{\overline{n-1}.\ n} + 2.\ 4\frac{\overset{4}{\dot{l}}}{1\ -\ -\ 4}\times\frac{\overset{n-4}{\dot{l}}}{1\ -\ -\ \overline{n-4}}\times\frac{\overline{n-4}}{\overline{n-1}.\ n} + \&c.$$

when $\frac{n}{2}$ is odd to be continued to $\frac{n-2}{4}$ terms. When $\frac{n}{2}$ is even, the laſt term is $\frac{\overset{\frac{n}{2}}{\dot{l}}}{1.\ 2^2\ -\ -\ n^2}\times\frac{\frac{n}{2}\times\frac{n}{2}}{\overline{n-1}.\ n}$.

Whence taking $n = 4,\ 6,\ 8$, &c.

$$l = \frac{\ddot{l}}{2} + \frac{l^4}{1.\ 2.\ 3.\ 4} +, \&c. = \frac{a^2}{1.2} + \frac{a^4}{1.3.4} + \frac{a^6}{1.3.3.5} + \frac{17a}{1.3.3.5.7.8} + \frac{31a^{10}}{1.3.3.5.7.9.5} + \&c.$$

Example IV. Having given the logarithmic ſecant of 45° to the modulus radius $(r) = s$ and the logarithmic ſecant of any arch $a = s + l$ to find a in a ſeries aſcending by the powers of l.

Solution. Let $n =$ the ſecant of a. Then by the circle $\dot{a} = \frac{\dot{n}\,r^2}{n\sqrt{n^2 - r^2}}$ alſo becauſe $s + l =$ log. n to the modulus r, $\dot{l} = \frac{r\,\dot{n}}{n}$ $\therefore \dot{a} = \frac{r\,\dot{l}}{\sqrt{n^2 - r^2}}$ or (A) $\dot{a}^2 \times \overline{n^2 - r^2} = r^2\,\dot{l}^2$. To facilitate the computations of $\dot{a}$, $\ddot{a}$, $\overset{3}{\dot{a}}$, &c. from the equat. A, when $l = o$, $a =$ the arc of 45, and when alſo $n^2 = 2\,r^2$, let $\dot{a}^2 = x^2 =$ B. Now becauſe $\dot{n} = \frac{n\,\dot{l}}{r}$.

1ſt. Fluxion of $n^2 = 2\,n\,\dot{n} = \frac{2\,\dot{l}\,n^2}{r} = 2^2\,r\,l$, ſubſtituting for l, $\dot{l}$.

2d. Fluxion of $n^2 = \frac{2\,l}{r} \times \frac{2\,l\,n^2}{r} = 2^3\,l^2$.

— — — — — — —

m^{th} Fluxion of $n^2 = \frac{2^{m+1}\,l^{m}}{r^{m-2}}$

WHENCE from the equation (A) $\mathrm{B} \times \overline{n^2 - r^2} = r^2 \dot{l}^2$, we have by Frob. 2.

$(\dot{\mathrm{A}})$ $\dot{\mathrm{B}} r^2 + \mathrm{B}\, 2^2 r l = 0$ or $\dot{\mathrm{B}} = -\frac{4 l^3}{r}$.

$(\overset{2}{\dot{\mathrm{A}}})$ $\ddot{\mathrm{B}} r^2 + 2 \dot{\mathrm{B}}\, 2_2\, r l + \mathrm{B}\, 2^3\, l^2 = 0$ or $\ddot{\mathrm{B}} = \frac{24 l^4}{r^2}$.

$(\overset{3}{\dot{\mathrm{A}}})$ $\dddot{\mathrm{B}} r^2 + 3 \ddot{\mathrm{B}}\, 2^2 r l + 3 \dot{\mathrm{B}}\, 2^2 r l_2 + \mathrm{B} \frac{2^4 l^3}{r} = 0$ or $\overset{3}{\dot{\mathrm{B}}} = -208 \frac{l^5}{r^5}$.

$(\overset{4}{\dot{\mathrm{A}}})$ $\overset{4}{\dot{\mathrm{B}}} r^2 + 4 \overset{3}{\dot{\mathrm{B}}}\, 2^2 r l + 6 \overset{2}{\dot{\mathrm{B}}}\, 2^3 l^2 + 4 \dot{\mathrm{B}} \frac{2^4 l}{r} + \mathrm{B} \frac{2^5 l^4}{r^2} = 0$ or $\overset{4}{\dot{\mathrm{B}}} = 2400 \frac{l^6}{r^4}$.

$(\overset{5}{\dot{\mathrm{A}}})$ $\overset{5}{\dot{\mathrm{B}}} r^2 + 5 \overset{4}{\dot{\mathrm{B}}}\, 2^2 r l^2 + 10 \overset{3}{\dot{\mathrm{B}}}\, 2^2 l^2 + 10 \overset{2}{\dot{\mathrm{B}}} \frac{2^4 l^3}{r} + 5 \dot{\mathrm{B}} \frac{2^5 l^4}{r^2} + \mathrm{B} \frac{2^6 l_5}{r^3} = 0$ or $\overset{5}{\dot{\mathrm{B}}} = -\frac{34624\, l^7}{r^5}$.

&c.

Now when $a = 45^\circ$, $x = \dot{a} = \dot{l}$, whence taking the fluxions of the equation $x^2 = \mathrm{B}$ by Prob. 1, and fubſtituting for $\dot{l}$, l, we get

$$2 l \dot{x} = \dot{\mathrm{B}} = -\frac{4 l^3}{r}, \text{ or } \ddot{a} = \dot{x} = -\frac{2 l^2}{r}.$$

$2 l \ddot{x}$

$$2l\ddot{x} + 2\dot{x}^2 = \ddot{B} = \frac{24l^4}{r^2} \text{ or } \overset{3}{\dot{a}} = \ddot{x} = \frac{8l^3}{r^2}$$

$$2l\overset{3}{\dot{x}} + 2.3\dot{x}\ddot{x} = \overset{3}{\dot{B}} = -\frac{208l^5}{r^3} \text{ or } \overset{4}{\dot{a}} = \overset{3}{\dot{x}} = -\frac{56l^4}{r^3}$$

$$2l\overset{4}{\dot{x}} + \begin{array}{l} 2.3\overset{2}{\dot{x}}{}^2 \\ 2.4\overset{3}{\dot{x}}\dot{x} \end{array} = \overset{4}{\dot{B}} = \frac{2400l^6}{r^4} \text{ or } \overset{5}{\dot{a}} = \overset{4}{\dot{x}} = \frac{560l^5}{r_4}$$

$$2l\dot{x}^5 + \begin{array}{l} 2.5x\overset{4}{\dot{x}} \\ 2.10\ddot{x}\overset{3}{\dot{x}} \end{array} = \overset{5}{\dot{B}} = -34624 \text{ or } \overset{6}{\dot{a}} = \overset{5}{\dot{x}} = -\frac{7232l^6}{r_5}$$

&c. &c.

Hence while l by flowing from o becomes l, a from a ſemiquadrant becomes =

$$\text{a ſemiquadr.} + l - \frac{l^2}{r} + \frac{4l^3}{3r^2} - \frac{7l^4}{3r^3} + \frac{14l^5}{3r^4} - \frac{452l^6}{45r^5} + \&c.$$

The two laſt examples are ſeries of Gregory's from the Comm. Epiſt. For an account and different methods of inveſtigating them ſee Scrip. Log. Vol. III. preface and pages 443, &c. 480, &c.

Example V. To expand the multinomial, $\overline{a + bz + cz^2 + dz^3 + \&c.}|^{n}$ where n is of any denomination whole, negative or fractional. (De Moivre Miſcell. Analytica, p. 87).

LET $A = B^n = \overline{a + bz + cz^2 + dz^3 + \&c.}^{\,n}$ Then making $z = o$, and ſubſtituting for $\dot{z}$, z

$$\begin{array}{lcl} & & \dot{B} = bz \\ \ddot{B} = 1.\,2\, cz^2 & \text{or} & \dfrac{\ddot{B}}{1.\,2} = cz^2 \\ \dddot{B} = 1.\,2.\,3\, dz_3 & & \dfrac{\overset{3}{\dot{B}}}{1.\,2.\,3} = dz^3 \\ \overset{4}{\dot{B}} = 1.\,2.\,3.\,4\, ez^4 & & \dfrac{\overset{4}{\dot{B}}}{1 \ldots 4} = ez^4 \\ \&c.\ \&c. & & \&c. \end{array}$$

WHENCE from the equation $A = B^n$ we have by Cor. Prob. 1.

$$\begin{array}{l} \dot{A} = na^{n-1}\, bz \\ \ddot{A} = 2.\, na^{n-2}\, cz^2 + n.\,\overline{n-1}\, a^{n-2}\, b^2 z^2 \\ \overset{3}{\dot{A}} = 3.\,2\, na^{n-1}\, dz_3 + 3.\,2.\,n.\,\overline{n-1}\, a^{n-2}\, bcz + n.\,\overline{n-1}.\,\overline{n-2}\, b^3 z^3 \\ \&c. \qquad\qquad\qquad \&c. \end{array}$$

$$\begin{aligned} \therefore A = \overset{n}{a} + \dot{A} + \frac{\ddot{A}}{1.\,2} + \&c. = a + na^{n-1}\, bz &+ \frac{n.\,\overline{n-1}}{1.\quad 2}\, a^{n-2}\, b^2 z^2 + \frac{n.\,\overline{n-1}.\,\overline{n-2}}{1.\quad 2 \quad 3}\, a^{n-3}\, b^3 z^3 + \&c. \\ n.\quad a^{n-1}\, c\, z^2 &+ n.\,\overline{n-1}\quad a^{n-2}\, bcz^3 + \&c. \\ &+ n\, a^{n-1}.\quad d\quad z^3 + \&c. \\ &\quad \&c. \qquad\qquad \&c. \end{aligned}$$

THE law of continuation as far as regards the products of z, and its coefficients a, b, c, &c. is evident from Prob. 1. and agrees with that given by De Moivre. The law of the coefficients of theſe products is alſo immediately derived. For let $a^{n-p}\,b^{q}\,c^{r}\,d^{s}$ be any product, then becauſe it occurs in the terms $\overset{q+2r+3s}{\dot{\mathrm{A}}}$, it follows from Taylor's Theorem and Prob. 1. that its coefficient is

$$\frac{1}{1.2.3\cdots\overline{q+2r+3s}}\times n.\overline{n-1}.\overline{n-2}\cdots\overline{n-p-1}\times\frac{\overline{q+2r+3s}\times\overline{q+2r+3s-1}\times\cdots 1}{q\times\overline{q-1}\times\cdots 1\times r\overline{r-1}\;1\times s\times\overline{s-1}\times\cdots 1}$$

$$=\frac{n.\overline{n-1}.\overline{n-2}\cdots\overline{n-p-1}\quad(p\text{ or }\overline{q+r+s}\text{ terms})}{q\times\overline{q-1}\times\cdots 1\times r\times\overline{r-1}\times\cdots 1\times s\times\overline{s-1}\times\cdots 1}.$$ The ſame as has been demonſtrated by De Moivre for integral values of n.

EXAMPLE VI. From the equation $(m)\ az+b\overset{2}{z}+c\overset{3}{z}+d\overset{4}{z}+$ &c. $=gy+hy^2+iy^3+ky^4+$ &c. to find the value of z when z and y begin together.

SOLUTION. Taking by Cor. Prob. 1. the ſucceſſive fluxions when z and $y=0$, and y is ſubſtituted for $\dot{y}$.

$(\dot{m})\ a\dot{z}=gy$ or $z=\dfrac{gy}{a}=\mathrm{A}y$ putting $\mathrm{A}=\dfrac{g}{a}$

$(\ddot{m})\ a\overset{2}{\dot{z}}+1.2\,\mathrm{A}^2\,by^2=1.2\,hy^2$ or $\dfrac{\overset{2}{\dot{z}}}{1.2}=\dfrac{h-b\mathrm{A}^2}{a}y^2=\mathrm{B}y^2$

$$\overset{3}{(\dot{m})}\ a\overset{3}{\dot{z}} + 1.2.3.2.1\,\mathrm{AB}by^2 + 1.2.3\,\mathrm{A}_3\,cy^3 = 1.2.3\,iy_3$$

or $$\frac{\ddot{z}}{1.2.3} = \frac{i - 1.2\,\mathrm{AB}b - \mathrm{A}^3c}{a}y^3 = \mathrm{C}y^3$$

$$\overset{4}{(\dot{m})}\ a\overset{4}{\dot{z}} + 1.2.4.3.2.1\,\mathrm{AC}by^4 + 1.2.4.3\,\mathrm{B}^2by^4 + 1.2.3.4.3\,\mathrm{A}^2\mathrm{B}cy^4 = 4.3.2.1\,hy^4$$ or $$\frac{\overset{4}{\dot{z}}}{1.2.3.4} = \frac{k - 1.2\,\mathrm{AC}b - \mathrm{B}^2b - 3\,\mathrm{A}^2\mathrm{B}c}{a}y^4 = \mathrm{D}y^4$$

&c. &c.

or $$z = \frac{g}{a}y + \frac{h - b\mathrm{A}^2}{a}y^2 + \frac{i - 1.2\,\mathrm{AB}b - \mathrm{A}\overset{3}{c}}{a}\overset{3}{y} + \frac{k - 1.2\,\mathrm{AC}b - \mathrm{B}^2b - 3\,\mathrm{A}^2\mathrm{BC}}{a}y^4 + \text{\&c}$$ A, B, C, &c. being the coefficients of the preceding terms.

The laws of continuation are readily derived by help of Prob. 1. for calling the exponent of a, 1 of b, 2 &c. and of A, 1 of B, 2 &c. the coefficient of y^m is a fraction the denominator of which is a, and the numerator the difference between the coefficient of y^m in the given equation, and the ſum of products of the capital and ſmall letters with numeral coefficients derived by the following laws:

1. To the ſmall letter the exponent of which is n are to be affixed n capital letters, ſo that the ſum of their exponents ſhall be m: this is to be done as often as poſſible with each ſmall letter.

2. THE numeral coefficient of any product $A^p B^q C^r =$

$$\frac{1.2 \cdots \overline{p+q+r} \times \overline{p+2q+3r} \times \overline{p+2q+3r}-1 \cdots 1}{1.2 \cdots \overline{p+2q+3r} \times p.\overline{p-1} \times \cdots 1 \times q \times \overline{q-1} \times \overline{q-2} \cdots 1 r \times \overline{r-1} - 1}$$

$$= \frac{1.2 \cdots\cdots p\overline{+q+r}}{p.\overline{p-1} \cdots 1 + q \times \overline{q-1} \cdots 1 \times r \times \overline{r-1} \cdots 1}$$ = the number of permutations of A A A (*p* things) B B (*q*) C C (*r*). Theſe laws of continuation are the ſame as ſtated by De Moivre*, and deduced by him from the application of the multinomial theorem.

EXAMPLE VII. From the mean anomaly of a planet to deduce the eccentric anomaly in a ſeries aſcending by the powers of the excentricity. Fig.

SOLUTION. Let A P B be the ſemi-elliptic orbit deſcribed about the focus S and centre C, and P the planet: then drawing R P D perp. to A B meeting the circle deſcribed on the diameter A B, the $\angle$ A C R will be the eccentric anomaly. Let the mean anomaly $= m$ (rad. $= 1$) the eccentric anomaly $= c$, A C $= 1$, and C S the eccentricity $= e$. Then m: circumference : : area A S P : area of the ellipſe : : area A S R : area of the circle $\because$ becauſe C R $= 1$, $m = 2$ area A S R $= 2$ A C R $+ 2$ C S R $=$ B R $\times$ C R $+$ C S $\times$ D R $= c + e\,s,\,c$ or $m = c + e\,s,\,c$.

LET

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* Philoſophical Tranſactions, Vol. XX. p. 190.

LET the ſucceſſive fluxions of this equation be taken by Prob. 2 and 3, when $e = o$, and $c = m$

$$\dot c + \dot e\, s, m = o$$

$$\ddot c + 2\,\dot e\,\dot c\; cs, m = o$$

$$\overset{3}{\dot c} + 3\,\dot e\,\ddot c\; cs, m - 3\,\dot e\,\dot c^2\, s, m = o$$

$$\overset{4}{\dot c} + 4\,\dot e\,\dot c^3\; cs, m - 4\,\dot e .\, 3\,\overset{2}{\dot e}\,\dot c\, s, m - 4\,\dot e\,\overset{3}{\dot c}\; cs, m = o$$

whence ſubſtituting for $\dot e$, e

$$\dot c = - e\, s, m.\ \ddot c = - 2\,\dot c\, c\; cs, m = 2^2\, s, m \times cs, m = e^2\, s, 2\,m$$

$$\overset{3}{\dot c} = - 3\,e^2\,\overset{2}{\dot c}\; cs, m + 3\,e\,\dot c^2\, s, m = - 3\,e^3 \times \overline{s, 2\,m \times cs, m - s^3, m} =$$

$$- \tfrac{3}{4}\, e^3 \times \overline{3\,s, 3\,m - s, m}.$$

$$\overset{4}{\dot c} = - 4\,\dot e \times \overset{3}{\dot c}\; cs, m - 3\,\overset{2}{\dot c}\,\dot c s, m - \overset{3}{\dot c}\, cs, m = 4\,e^4 \times \overline{2\,s, 4\,m - s, 2\,m}$$

&c. &c.

$$\because c = m + \dot c + \frac{\ddot c}{1.2} + \&c. = m - e\, s, m + \frac{e^2}{1.2}\, s, 2\,m - \frac{e_3}{1.2.4} \times$$

$$\overline{3\,s, 3\,m - s, m} + \frac{c^4}{1.2.3} \times \overline{2\,s, 4\,m - s, 2\,m} + \&c.$$

THIS ſeries is in effect the ſame as the ſeries given by Keil, but is much better adapted for computation, and beſides has the advantage of being applicable to phyſical aſtronomy; which the ſeries of Keil is not.*

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* M. De la Grange has given a moſt elegant theorem for expreſſing in a ſeries aſcending by the powers of t any function of x, when $x =$ any function of $u + t\,X$, X being a function of x. By help of his beautiful theorem, the value of c is immediately deduced from the equation $m = c + e\, s, c$. But as the theorem is only adapted to equations of that particular form, it appears equally eligible to deduce the value of c by the above method, becauſe including the demonſtration the method of De la Grange is not ſhorter. See Couſin's Aſtro. Phyſ. Art. 20, page 15.

Example VIII. From the mean anomaly of a planet to deduce the true anomaly in a ſeries aſcending by the powers of the eccentricity.

Solution. Let the ſemi-axis major $AC = 1$. The eccentricity $CS = e$, the anomaly $AST = a$, $m =$ the correſponding mean anomaly meaſured in the circle the rad. of which $= 1$, and the periphery P. Then as the areas are proportional to the times, and therefore to the mean anomalies: Fig.

Flux. area AST : area of the ellipſe :: $\dot{m} : P \therefore \frac{1}{2} ST^2 \times$ flux. $\angle AST\ (\dot{a}) =$ flux. area $AST = \frac{\dot{m} \times \text{area of the ellipſe}}{P} = \dot{m} \times \frac{1}{2}\sqrt{1-e_2}$ or $\dot{a} \times ST^2 = \dot{m}\sqrt{1-e^2}$. But by the prop. of the ellipſe $ST = \frac{1-e^2}{1-e\,cs, a}$. Hence $\dot{a} \times \frac{\overline{1-e^2}|^{\frac{3}{2}}}{\overline{1-e\,cs, a}|^2} = \dot{m}$ or $\overline{1-e^2}|^{\frac{3}{2}} \times \dot{a} \times \overline{1+2e\,cs, a+3e^2\,cs^2, a+4e^3\,cs^3, a+ \&c.} = \dot{m}$. Let $A =$ fl $\dot{a}\,cs\,a$, $B =$ fl $\dot{a}\,cs^2, a$, $C =$ fl $\dot{a}\,cs^3, a$, &c. and $L = \overline{1-e^2}|^{-\frac{3}{2}}$. Then $a + 2eA + 3e^2B + 4e^3C + \&c. = Lm$. From this equation the ſeries is to be deduced by ſucceſſively taking its fluxions by Prob. 2. making e flow uniformly &c. $e = o$

1. $\dot{a} + 2\dot{e}A = \dot{L}m = o$

2. $\ddot{a} + 2.\,2\dot{e}\dot{A} + 3.\,2\dot{e}^2B = 3\dot{e}^2m$

3. $\overset{3}{a}$

$3.\,\overset{3}{\dot{a}} + 3.\,2\,\dot{e}\,\ddot{A} + 3.\,3.\,2\,\dot{e}^2\,\dot{B} + 4.\,3.\,2\,\dot{e}^3\,C = \overset{3}{\dot{L}}.m = 0$

$4.\,\overset{4}{\dot{a}} + 4.\,2\,\dot{e}\,\overset{3}{\dot{A}} + 6.\,3.\,2\,\dot{e}^2\,\overset{2}{\dot{B}} + 4.\,4\,3.\,2\,\dot{e}^3\,\dot{C} + 5.\,4.\,3.\,2\,\dot{e}^4\,D = \dot{L}.m = 45\,\dot{e}^4$

&c. &c.

Now ſince $A = s, a$

By Prob. 3. $\dot{A} = \dot{a}\,cs, a$ $\qquad B = \frac{1}{2}a + \frac{1}{4}s, 2a$

$\ddot{A} = \ddot{a}\,cs, a - \dot{a}^2 s, a$ $\qquad \dot{B} = \frac{1}{2}\dot{a} + \frac{1}{2}\dot{a}\,cs, 2a$

$\overset{3}{\dot{A}} = \overset{3}{\dot{a}}\,cs, a - 3\ddot{a}\dot{a}\,s, a - \dot{a}^3 cs, a$ $\qquad \ddot{B} = \frac{1}{2}\ddot{a} + \frac{1}{2}\ddot{a}\,cs, 2a - \dot{a}^2 s, 2a$

$C = \frac{3}{4}s, a + \frac{1}{12}s, 3a$ $\qquad$ and $D = \frac{3}{8}a + \frac{1}{4}s, 2a + \frac{1}{32}$

$\dot{C} = \frac{3}{4}\dot{a}\,cs\,a + \frac{1}{4}\dot{a}\,cs, 3a$

&c. &c.

Let theſe values be ſubſtituted in the above equations, and we deduce making $\dot{e} = e$ from the

1ſt. Equat. $\dot{a} = -2e\,s, m$

2d. $\ddot{a} = \frac{5}{2}e^2 s, 2m$

3d. $\overset{3}{\dot{a}} = -e^3 \times \overline{\frac{13}{2}s, 3m + \frac{3}{2}s, m}$

4th. $\overset{4}{\dot{a}} = e^4 \times \overline{\frac{103}{4}s, 4m + 11\,s, 2m}$

&c. &c.

Whence $a = m + \left.\begin{matrix}2e\\ \frac{1}{4}e^3\end{matrix}\right\} s, m \left.\begin{matrix}+\frac{5}{4}e^2\\ -\frac{11}{24}e^4\end{matrix}\right\} s, 2m - \frac{13}{12}e^3 s, 3m + \frac{103}{96}e^4 s, 4m +$ &c.

The ſecond power of the eccentricity or two terms of the ſeries will be ſufficient for the orbits of the Earth and Venus. The third power of the eccentricity or three terms for Jupiter, Saturn, and the Georgium Sidus, and four for Mars. But ſix or ſeven are neceſſary for Mercury. It is more tedious than difficult to continue this ſeries to a greater number of terms. The above ſolution of this uſeful and celebrated problem, beſides being direct is greatly ſhorter than any, before given: Even than the method of Cagnioli, given by De La Lande, in the third volume of his Aſtronomy, edition 1792, where the ſeries is continued to the ninth power of the eccentricity. By a very ingenious artifice there given the ſolution by indeterminate coefficients is very conſiderably ſhortened. The legality of that artifice might however be juſtly doubted, and the truth of the concluſion deduced ſuſpected, unleſs verified by other methods.

Example IX. From the equation $cx^n\dot{x} + y\dot{x} = a\dot{y}$ to find y by a ſeries aſcending by the powers of x, n being a whole poſitive number (Simpſon's Fluxions, Vol. II. 293).

Solution. When $x = o$, let $y = Y$. Then taking the fluxions of the given equation when $x = o$, and x flows uniformly.

1ſt. $\dot{y}\dot{x} = a\ddot{y}$

2d. $\ddot{y}\dot{x} = a\overset{3.}{y}$

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$n^{th}.\ n.\ \overline{n-1}\ ---\ 1\,c\,\overset{n+1}{\dot{x}} + \overset{n}{\dot{y}}\,\dot{x} = a\overset{n+1}{\dot{y}}$

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$m^{th}.$ - - - - - - $\overset{m}{\dot{y}}\,\dot{x} = a\overset{m+1}{\dot{y}}$

&c. &c.

Then becauſe when $x = 0$, and $y = Y$, $\dot{y} = Y\dot{x}$ we immediately deduce

$$\ddot{y} = \frac{Y\dot{x}^2}{a^2},\ \overset{3}{\dot{y}} = \frac{Y\dot{x}^3}{a^3} ----\overset{n+1}{\dot{y}} = n.\overline{n-1}--1\frac{c\overset{n+1}{\dot{x}}}{a} + \frac{Y\dot{x}^n}{\dot{a}^n},\ \overset{m+1}{\dot{y}} = \frac{\overset{m}{\dot{y}}\dot{x}}{\dot{a}},\ \&c.$$

$\because$ ſubſtituting for $\dot{x}$, x; $y = Y + Y \times \overline{\frac{x}{a} + \frac{x^2}{1.2a^2} + ----\frac{x^n}{1.2.-na^n}}$

$$\overline{+\frac{x^{n+1}}{1.2.\overline{n+1}a^n} + \&c.} + \frac{cx^{n+1}}{\overline{n+1}a} + \frac{cx^{n+2}}{\overline{n+1}.\overline{n+2}a_2} + \&c.$$

This example was given to remark that ſometimes by this method we may derive a general ſolution from the particular one. For although the above ſolution is only a particular one viz. when x is ſuch that the ſeries will converge, yet becauſe we know that $1 + \frac{x}{a} + \frac{x^2}{1.2a_2} + \&c.$ = no. the hyp. log. of which is $\frac{x}{a}$, and alſo becauſe $\frac{cx^{n+1}}{\overline{n+1}.a} + \frac{cx^{n+2}}{\overline{n+1}.\overline{n+2}a_2} + \&c. =$

$$ca^n \times 1.2--n \times \overline{1 + \frac{x}{a} + ---\frac{x^{n+1}}{1.2--\overline{n+1}a^{n+1}} + \&c. - 1 + \frac{x}{a} + --\frac{x_n}{1.2.--na_n}}$$

$$\frac{x}{1.\,2 - - - na} = ca_n \times 1.\,2 - - - n \times \text{no. hy. log.}\ \overline{\frac{x}{a} - 1 + \frac{x}{a} + - -}$$

$$\overline{\frac{x}{1.\,2. - - na'}};\ \text{if M} = \text{no. hyp. log. of which is } 1,\ y = \text{Y M}^{\frac{x}{a}} + 1.\,2 - -$$

$$- nca^{n} \times \text{M}^{\frac{x}{a}} - \overline{1.\,2 - - - nca^{n} + 1.\,2 - - - - nca^{n-1}\,x + 3.\,4 - -}$$

$\overline{nca^{n-2}\,x^{2}} + - - - - + cx^{n}$ is the general equation of the fluents.

As the above examples have conſiderably extended the length of this tract, the ſubject ſhall be concluded by a few obſervations.

The Theorem of Taylor may be more generally expreſſed, for if z be a quantity compoſed of two or more independent quantities x, y, v, &c. then while x, y, v, &c. by flowing uniformly become $x + \dot{x}$, $y + \dot{y}$, $v + \dot{v}$, &c. z will become $\dot{z} + \frac{\ddot{z}}{1.\,2} +$ &c. There can be no difficulty in applying what has been before done to caſes of this kind. It may be worthy of remark, however, that by this method when fluxions are ſuch that the fluents are expreſſed in integral powers, they may be found *a priori*: for if z be a function of x, y, &c. where x, y, &c. are independent quantities, and Z the value of z when x, y, &c. $= o$, then becauſe $z = \text{Z} + \dot{z} + \frac{\ddot{z}}{1.\,2} +$ &c. and becauſe $\frac{\ddot{z}}{1.\,2}$, &c. are derived from $\dot{z}$ by taking the fluxions, making $\dot{x}, \dot{y}$, &c. conſtant, it follows that z may be deduced from $\dot{z}$ by taking the ſucceſſive fluxions of $\dot{z}$ by the former problems.

Examples. The fluent of $\overset{n}{x}\,\dot{x} = \text{Cor.} + \frac{n.\overline{n-1}\,---\,1}{1.2\,---\,\overline{n+1}}\,x^{n+1} = \frac{x^{n}}{n+1} + \text{Cor.}$ The fluent of $3\,x^2 y \dot{x} + x^3 \dot{y} + 2\,xy^2 \dot{x} + 2\,x^2 y \dot{y} =$ (taking x and $y = o$, and ſubſtituting for $\dot{x}$ and $\dot{y}$, x and y) $\frac{3.3.2\,x^3 y + 3.2\,x^3 y + 2.3.2\,x^2 y^2 + 2.3.2\,x^2 y^2}{1.\ 2.\ 3.\ 4} = x^3 y + x^2 y^2.$

The fourth example when n is odd is an inſtance of finding fluents *a priori* by this method. If $\dot{x} = Y\dot{y}$, where Y is an algebraic function of y, then by common algebra reducing this equation to integral values, and taking the fluxions *particularly* by the former rules, it will be known whether x the fluent can be had in finite terms; in ſome caſes, very readily, in many, however, the difficulty will greatly exceed the inverſe method, but this difficulty may be probably obviated by given the ſubject that attention it ſeems to deſerve.

But it ought to be remarked when there are two or more independent variable quantities, that the given fluxion muſt be poſſible, that is, muſt have originated from a fluent. Thus for inſtance $y\dot{x}$ is not a poſſible fluxion, for it cannot have originated from any flowing quantity wherein x and y are independent.

THE above method may alſo be applied with conſiderable advantage to the finite variations of ſpherical triangles, and in many inſtances ſeries may be deduced more convenient in aſtronomical computations than the theorems for finite differences.